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Finite strain estimation using the mean radial length of elliptical objects with bootstrap confidence intervals

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Abstract

A new method for calculating finite sectional strain from distributions of elliptical objects is presented. The only assumptions required are that before deformation (1) long axis orientations are uniformly distributed and (2) the distribution of axial ratios is independent of orientation. Importantly, an estimate of the orientation of the long axis of the strain ellipse is not required before the method can be applied. The method is based on the conceptually simple fact that the mean radial length of a set of uniformly oriented ellipses in the unstrained state equates to that of a circle, so that after strain, the mean radial length evaluates to the strain ellipse. Errors associated with the method are calculated from the bootstrap, and a simulation study verifies both the applicability of the new method for finite strain estimation and the accuracy of errors calculated with the bootstrap. The method is applied to a large set of sandstone quartz clast data from the Irish Variscides, whilst cross-checking of results with those from established methods also validates the approach taken.

Keywords: Finite strain; Mean radial length; Bootstrap; Ellipse; Confidence interval

1. Introduction

Ramsay (1967) first introduced the idea of determining finite strain from randomly oriented populations of deformed elliptical objects, which was further developed by Dunnet (1969) and extended using graphical methods by Elliott (1970) and Dunnet and Siddans (1971) to include non-random initial distributions. This technique has become known as the R_f/ϕ method. All of the above methods require non-repeatable, subjective application of graphical techniques, which is not ideal. Matthews et al. (1974) developed a purely algebraic method for strain ellipse determination enabling statistical treatment of errors and precision, though the orientation of the principal strain axis needs to be independently estimated prior to calculating the finite strain ellipse. This approach has proved unpopular.Using the usual assumptions of an initial random distribution and homogenous deformation together with the matrix, Shimamoto and Ikeda (1976) developed a simple algebraic method of analysis, whose only limitation is that it requires evaluation of the eigenvalues of a matrix. Robin (1977) describes the theory and application of a strain analysis method for objects of any shape, which may be applied to elliptical objects, but requires the orientation of the principal strain axes to be independently determined. Le Theoff (1979) demonstrated the applicability of the method of Dunnet (1969) to non-coaxial strain states and De Paor (1980) showed that elliptical distributions cannot be separated into an initial non-random part and a tectonic part. By linearising the equations of Dunnet (1969), Yu and Zheng (1984) took a least squares approach to analysing elliptical data. However, Mulchrone and Meere (2001) have cast doubt on the accuracy of this approach, probably due to the resulting data distribution being non-Gaussian, a fundamental assumption of linear least-squares regression.

In a somewhat separate research stream several workers have investigated the properties of strained distributions of ellipses. Lisle (1977a) investigated measures of strain such as the arithmetic, geometric and harmonic means and Lisle (1977b) introduced the theta curve method and considered the influence of pressure solution. Borradaile (1976) introduced the minimum χ^2 method, which involves destraining an ellipse population about an independently determined finite strain axis until the distribution becomes most uniform and Peach and Lisle (1979) developed a

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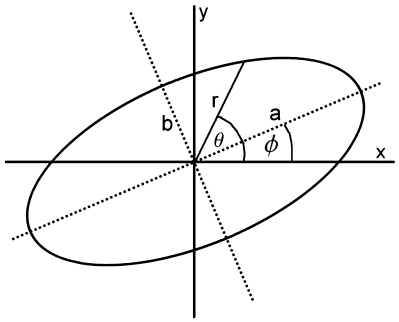

Fig. 1. An ellipse with long axis of length a and short axis of length b, whose long axis makes an angle ϕ with the positive x-axis. θ and r are polar coordinates for a point on the ellipse.

program for this method. Borradaile (1987) discussed the impact of an initial fabric on strain analysis and Borradaile and McArthur (1991) compared the methods of Yu and Zheng (1984), Robin (1977) and Dunnet and Siddans (1971), illustrating the importance of geological observation in combination with each method.

In summary, the above research effort illustrates the importance of the analysis of ellipse distributions in determining finite strain for structural geologists. Why another method for analysis? The technique described in this paper has at least one of several advantages over previous approaches:

1. It is simple enough to be implemented in a spreadsheet application.
2. It is non-graphical.
3. It produces repeatable, objective results.
4. Confidence limits for the results are readily calculated.
5. An estimate of the orientation of the long axis of the strain ellipse is not a prerequisite.
6. It requires that the distribution of axial ratios is independent of orientation.

For example, Shimamoto and Ikeda (1976) requires an eigenvector/eigenvalue calculation, usually not included with standard spreadsheet applications. The methods of Matthews et al. (1974) and Robin (1977) require an independent estimate of the orientations of the principal axes of the strain ellipse. Robin (1977) suggests this estimate may be evaluated from analysis of the long axes of elliptical objects or by repeatedly applying the technique for different assumed orientations of the principal strain axes and taking the orientation which maximises the strain ratio. However, any error in estimating the orientation of the principal strain axes automatically propagates into the strain ratio calculation.

2. Derivation of the method

2.1. Assumptions

The method derived here for determination of finite strain is based upon the following basic assumptions:

1. The population of elliptical objects have the following characteristics in the unstrained state: (1) long axis orientation (ϕ) is a uniform random variable on the $[0,\pi]$ interval (Fisher, 1993, p. 43), i.e. it is equally probable that the long axis lies along any direction in the interval, and (2) the distribution of axial ratios (R) is independent of orientation, which means that R is an isotropic, random variable (note that the exact type of random variable does not need to be specified). For the sake of conciseness a population of elliptical objects with these prescribed characteristics is simply referred to as isotropic.
2. Deformation within the region of analysis can be considered homogeneous.
3. The viscosity contrast between ellipsoidal objects and surrounding matrix approaches unity.

Although the analysis may be applied to distributions which do not strictly adhere to the above assumptions, erroneous conclusions may be drawn if these factors are not taken into account. The assumptions presented above also apply to the R_f/ϕ method (Ramsay, 1967; Dunnet, 1969; Dunnet and Siddans, 1971; Lisle, 1977a,b), although the method has been extended to take account of deviations from these assumptions (Dunnet and Siddans, 1971).

2.2. Theoretical basis of the method

2.2.1. Introduction

It is shown below that the mean radial length of an isotropic distribution of ellipses is a constant independent of orientation, i.e. forms a circle. Therefore, if the ellipse distribution is subjected to homogenous deformation, the circle of mean radial length is transformed to the finite strain ellipse, in the same way as any other material circle is transformed.

2.2.2. The mean radial length of a random distribution of ellipses

A few basic concepts and related equations are introduced before proving that the mean radial length of a random distribution of ellipses is constant. The following discussion refers to a distribution of ellipses in the undeformed state and every effort is made to conform with existing notation and standard equations (e.g. Ramsay, 1967, pp. 65–66). However, some of the equations given below are probably unfamiliar but have been chosen to simplify the mathematics.

Fig. 1 illustrates an ellipse centred on the origin whose half axes are of length a and b, respectively, and whose long axis makes an angle ϕ with the x-axis. The equation of such

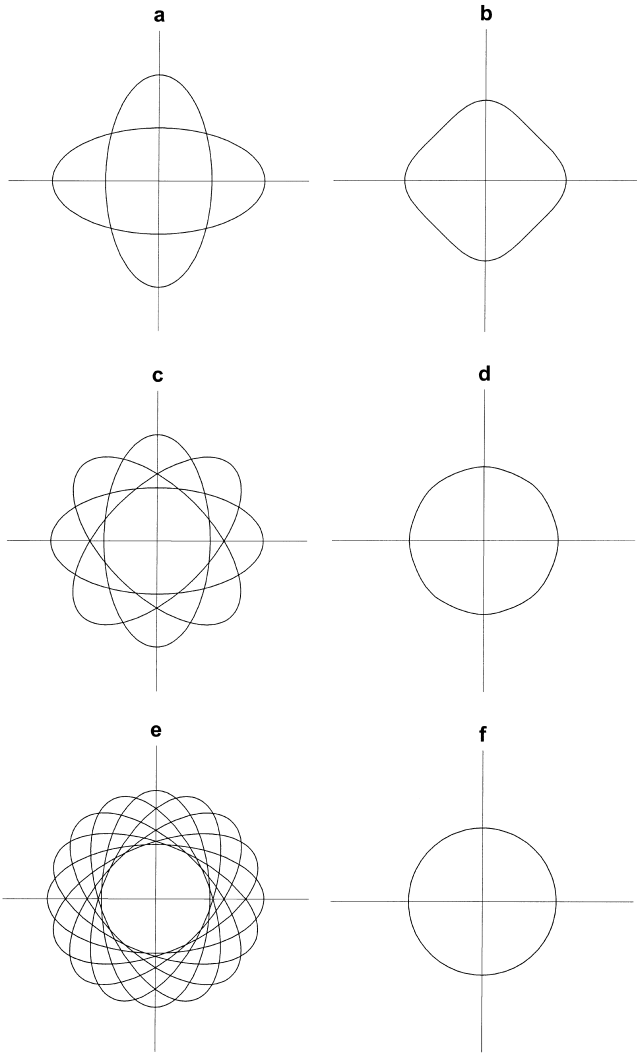

Fig. 2. Diagrams on the left (a, c and e) illustrate equally spaced ellipses having an axial ratio of two. Diagrams on the right (b, d and f) show the corresponding mean axial ratio for the ellipses shown on the left. In diagram (e) there are eight ellipses but the mean radial length is almost indistinguishable from a circle. Although this diagram illustrates the case for taking the mean of r, a similar behaviour is also exhibited for almost any other function of r.

an ellipse in polar co-ordinates (r,θ) is given by:

$$r = a\sqrt{\frac{1-e^2}{1-e^2\cos^2(\theta-\phi)}} \tag{1}$$

where θ varies from 0 to 2π and e, the eccentricity, is given by:

$$e = \sqrt{1-\frac{b^2}{a^2}} \tag{2}$$

and the vector $\mathbf{r}$ is the radial length, whose length is r, generated by joining the origin to each point on the ellipse. By noting that the axial ratio of an ellipse is $R = a/b$ and by scaling each ellipse such that its area equals that of the unit circle (i.e. π):

$$a = \sqrt{R} \tag{3}$$

$$b = \frac{1}{\sqrt{R}} \tag{4}$$

Eq. (1) becomes:

$$r = \sqrt{\frac{R}{R^2+(1-R^2)\cos^2(\theta-\phi)}} \tag{5}$$

Suppose the value of R and θ are kept fixed, then the mean radial length along θ is calculated using the integral (see Fraleigh, 1990, p. 303, for example):

$$\bar{r} = \frac{1}{\pi}\int_0^{\pi} r\,\mathrm{d}\phi \tag{6}$$

However, this is equivalent to:

$$\bar{r} = \frac{1}{\pi}\int_0^{\pi} r\,\mathrm{d}\theta \tag{7}$$

where R and ϕ are kept fixed. This is because both θ and ϕ are uniformly distributed and periodic in the range $[0,\pi]$ so that the values taken by $\theta-\phi$ in Eq. (5) are the same whether θ is fixed and ϕ varied or vice versa. Notice that the integrals are evaluated over the interval $[0,\pi]$. This is because geological data is usually axial in the sense of Mardia (1972) and Batschelet (1981). Evaluating the integral in Eq. (7) gives:

$$\bar{r} = \frac{2\sqrt{R}}{\pi}K\left(1-R^2\right) \tag{8}$$

where $K(m)$ is the complete elliptic integral of the first kind (Arfken and Weber, 1995, p. 333). $K(1-R^2)$ evaluates to a constant for particular values of R and therefore $\bar{r}$ is also constant.

This result implies that for a continuous distribution of ellipses, with long axis orientations uniformly distributed in the range $[0,\pi]$ and all with the same value of R, the mean radial length ($\bar{r}$) takes the same constant value for all θ, i.e. defines a circle (see Fig. 2). In addition, any function of $\bar{r}$ also defines a circle, as does the mean of any function of r. This is because any function of r will also permit the interchange of θ and ϕ, required for the proof given above. A similar result was proved previously by Shimamoto and Ikeda (1976) for a different set of parameters in terms of matrix algebra and integral calculus.

The result just established is now extended to the mean radial length of a distribution of ellipses whose orientations are uniformly distributed in $[0,\pi]$ and axial ratio R randomly distributed in $[R_{\min},R_{\max}]$, where $R_{\min}$ and $R_{\max}$ are arbitrary lower and upper limits, respectively. Let this distribution be denoted by $E[(0,\pi);\ (R_{\min},R_{\max})]$. $E[(0,\pi);\ (R_{\min},R_{\max})]$ is composed of infinitely many distributions of the form $E[(0,\pi);\ (R,R)]$ where R takes on all values in the interval $[R_{\min},R_{\max}]$. From the result above, $\bar{r}$ for each distribution $E[(0,\pi);\ (R,R)]$ is a constant

and therefore the average of $\bar{r}$ for all $E[(0,\pi);(R,R)]$ $(=E[(0,\pi);(R_{\min},R_{\max})])$ is also a constant. Notice that the distribution of axial ratios in $[R_{\min},R_{\max}]$ need not be uniform and that the method is valid for any distribution of R in $[R_{\min},R_{\max}]$ provided that distribution does not vary with orientation, i.e. provided R is an isotropic random variable. The crucial assumption here is that the orientations of the long axes are uniformly distributed in $[0,\pi]$.

The theoretical basis of the method developed below is that because $\bar{r}$ versus θ defines a circle in the undeformed state, it follows that after deformation this circle is transformed into an ellipse, and that the resulting ellipse has the same orientation (ϕ_s) and axial ratio as the finite strain ellipse (R_s).

2.2.3. Calculation of ϕ_s and R_s of the finite strain ellipse

The theory developed above is now extended to allow calculation of ϕ_s and R_s, the orientation of the long axis and axial ratio of the finite strain ellipse, respectively. It is assumed that in the undeformed state there is a collection of elliptical objects with long axes uniformly distributed in $[0,\pi]$ and isotropic, randomly distributed axial ratios in $[R_{\min},R_{\max}]$. To reflect that we are dealing with the deformed state each ellipse in the collection is denoted by E_i' with corresponding parameters r_i', ϕ_i' and R_i', where r_i' is radial length, ϕ_i' is long axis orientation, R_i' is axial ratio and i goes from 1 to n, the total number of elliptical objects. Thus for E_i' the radial length, r_i', is given by:

$$r_i' = \sqrt{\frac{R_i'}{R_i'2+(1-R_i'2)\cos^2(\theta-\phi_i')}} \tag{9}$$

In order to facilitate statistical analysis the equation is rearranged by squaring, inverting and applying trigonometrical identities to give:

$$l_i' = \frac{1}{r_i'2} = p_i' - m_i'\cos(2\phi_i')\cos(2\theta) - m_i'\sin(2\phi_i')\sin(2\theta) \tag{10}$$

where:

$$m_i' = \frac{1}{2}\left(R_i' - \frac{1}{R_i'}\right) \tag{11}$$

$$p_i' = \frac{1}{2}\left(R_i' + \frac{1}{R_i'}\right) \tag{12}$$

This is referred to as the linearised polar equation of an ellipse. Note that the theory given earlier demonstrates that the mean of any function of r in the undeformed state evaluates to a constant, so that calculating the strain ellipse from the linearised polar equation of an ellipse is valid.

The mean of l_i' is calculated by:

$$\bar{l'} = \sum_{i=1}^{n} l_i' = \frac{1}{n}\sum_{i=1}^{n} p_i' - \frac{1}{n}\sum_{i=1}^{n}[m_i'\cos(2\phi_i')]\cos(2\theta) - \frac{1}{n}\sum_{i=1}^{n}[m_i'\sin(2\phi_i')]\sin(2\theta) \tag{13}$$

and by defining:

$$q_s = \frac{1}{n}\sum_{i=1}^{n} p_i' \tag{14}$$

$$t_s = \frac{1}{n}\sum_{i=1}^{n} m_i'\cos(2\phi_i') \tag{15}$$

$$u_s = \frac{1}{n}\sum_{i=1}^{n} m_i'\sin(2\phi_i') \tag{16}$$

Eq. (13) reduces to:

$$\bar{l'} = q_s - t_s\cos(2\theta) - u_s\sin(2\theta) \tag{17}$$

Eq. (17) is the mean of the linearised polar ellipse equations and is exactly equivalent to the linearised polar equation of the strain ellipse:

$$l_s = p_s - m_s\cos(2\phi_s)\cos(2\theta) - m_s\sin(2\phi_s)\sin(2\theta) \tag{18}$$

where:

$$m_s = \frac{1}{2}\left(R_s - \frac{1}{R_s}\right) \tag{19}$$

$$p_s = \frac{1}{2}\left(R_s + \frac{1}{R_s}\right) \tag{20}$$

Therefore, given a natural set of n data (i.e. R_i' and ϕ_i', $i=1$ to n), q_s, t_s and u_s are readily calculated and ϕ_s and R_s may be estimated by equating coefficients in Eqs. (17) and (18) as follows:

$$\tan(2\phi_s) = \frac{u_s}{t_s} \tag{21}$$

$$R_s = \sqrt{\frac{q_s\cos(2\phi_s)+t_s}{q_s\cos(2\phi_s)-t_s}} \tag{22}$$

3. Error estimation using the Bootstrap

The Bootstrap is a general technique invented by Efron (1979) to construct approximate sampling distributions for complex statistical estimates. It is widely regarded as one of the key advances in the field of statistics in the past several decades. The estimates ϕ_s and R_s described above are certainly complex statistics—they involve highly non-linear functions of the underlying data. Even if the underlying data were described by a simple distribution, such as a Gaussian, the structure of the equations leading to ϕ_s and R_s means that the construction of an analytic description of the

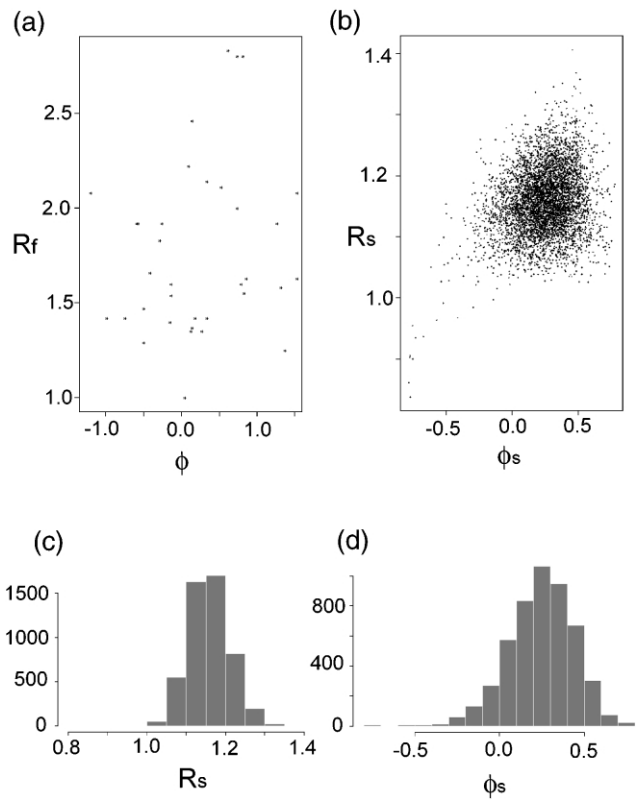

Fig. 3. Bootstrap analysis of 50 elliptical objects. (a) The original data set in (R_f, ϕ) space. Note that there are several duplicate values on this plot meaning that there are only 35 distinguishable points. (b) The joint bootstrap distribution for the calculated values of R_s and ϕ_s. The marginal distributions for R_s and ϕ_s are shown in (c) and (d), respectively.

sampling distribution of these estimates is intractable. This is a situation which is ideally suited for application of the Bootstrap.

The basis of the bootstrap is as follows: suppose an estimate $\hat{\theta}$ is obtained as a function of a set of n (possibly vector-valued) measurements $x_1,x_2,\ldots,x_n$

$$\hat{\theta} = \theta(x_1, x_2, \ldots, x_n)$$

Suppose the x_is can be regarded as a random sample from a distribution F. The sampling distribution of $\hat{\theta}$ is defined as the distribution of $\hat{\theta}$ over repeated random samples of size n from the distribution F. The bootstrap replaces the unknown population distribution F of the measurements by the empirical distribution F_n observed in the sample $x_1,x_2,\ldots,x_n$. The bootstrap sampling distribution is the distribution of $\hat{\theta}$ over repeated random samples of size n from the distribution F_n. Note that sampling from F_n is equivalent to random sampling with replacement from the data values $x_1,x_2,\ldots,x_n$. Denote a particular bootstrap sample by $x_1^*, x_2^*, \ldots, x_n^*$ and the corresponding estimate

$$\hat{\theta}^* = \theta(x_1^*, x_2^*, \ldots, x_n^*)$$

Sampling from F_n can be repeated an arbitrary number of times to obtain the bootstrap distribution for $\hat{\theta}^*$ to any desired degree of accuracy. There are many formal results demonstrating how the bootstrap provides accurate approximations to the true sampling distribution of the estimate. Basically, for large samples (n) the bootstrap distribution of $\hat{\theta}^* - \hat{\theta}$ tends to coincide with the sampling distribution of $\hat{\theta} - \theta$. Here θ is the true underlying parameter of interest. In particular, the standard deviation of the bootstrap distribution will approximate the standard deviation of the sampling distribution of $\hat{\theta}$ (i.e. the standard error of $\hat{\theta}$). Intuitively the bootstrap works because the empirical distribution, F_n, will tend to the underlying population distribution, F, as $n \to \infty$. For a recent account of the Bootstrap including a variety of refinements, see the monograph by Efron and Tibshirani (1993).

3.1. An illustration with finite strain data

In the context of finite strain estimation, the sample data are values of R_i and ϕ_i for the sample of n objects, thus $x_i = (R_i, \phi_i)$ for $i = 1,2,\ldots,n$. The estimate of interest is $\hat{\theta} = (\hat{R}_s, \hat{\phi}_s)$ where R_s and ϕ_s are defined in Eqs. (21) and (22). To illustrate the Bootstrap, a set of real data based on measurements of $n = 50$ elliptical objects was considered. This data comes from measurements of quartz clasts in sandstone sample 19bc of Meere (1995), see below for more detail on these samples. The data are shown in Fig. 3a. The parameters for the average strain ellipse were found to be $\hat{R}_s = 1.147$ and $\hat{\phi}_s = 0.261$ radians, respectively. The bootstrap distribution of $\hat{\theta}$ was determined based on 5000 samples. Each sample consisted of sampling a set of 50 objects with replacement from the original set of 50 elliptical objects.The joint bootstrap distribution of $\hat{R}_s$ and $\hat{\phi}_s$ as well as the marginal distributions are shown in Fig. 3b–d. The joint distribution is seen to have a wedge-shaped appearance and the marginal distributions are both skewed towards lower values. It should be noted that none of these characteristics are consistent with a Gaussian form for the sampling distribution.

The bootstrap estimates of the standard errors of R_s and ϕ_s are 0.051 and 0.199, respectively. Application of the percentile method (Efron, 1979) yields 95% confidence intervals for the average strain ellipse parameters. The intervals are (1.062,1.263) and (−0.167,0.605) for R_s and ϕ_s, respectively.

Robin and Torrance (1987) provided a method for estimating the confidence interval associated with calculating the axial ratio of the finite strain ellipse using the method of Robin (1977). Using these methods the axial ratio is found to be $\hat{R}_s = 1.13$ and the 95% confidence interval is (1.03,1.23), which compares favourably with the results from the bootstrap approach. Robin and Torrance (1987) were able to estimate the 95% confidence interval because the method of Robin (1977) essentially involves calculating a mean, so that the central-limit theorem applies.There are advantages and disadvantages to both the current method and that of Robin (1977) and Robin and Torrance (1987). Robin's method is simpler as is the method for calculating

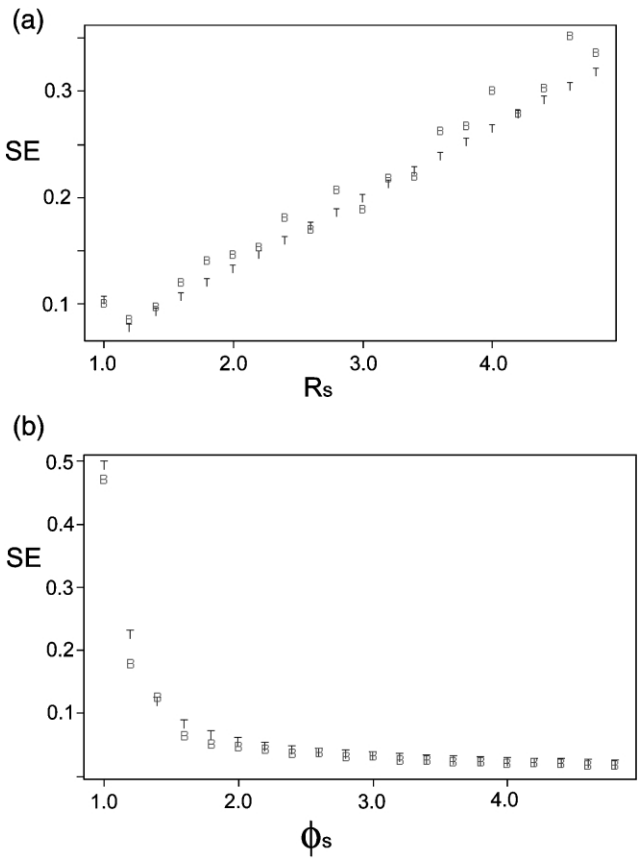

Fig. 4. Comparison of bootstrap standard error (SE) with true standard error calculated by simulation. B stands for a bootstrap value and T stands for true value estimated by simulation. Results for (a) R_s and (b) ϕ_s.

confidence interval, however, the method of Robin (1977) requires an independent estimate of the orientation of the finite strain axes and the uncertainty of this estimate is not included in the analysis of Robin and Torrance (1987). On the other hand the current bootstrap approach estimates the uncertainty in both R_s and ϕ_s but the method is slightly more complex and the bootstrap is computationally intensive.

4. Validation of the approach

4.1. A simulation study

In order to evaluate the performance of the Bootstrap approach in the context of finite strain estimation, a set of simulation studies was carried out. The general procedure used is as follows:

1. Initial data sets $x_i = (R_i, \phi_i)$, where $i = 1,\ldots,n$, were generated by choosing R_i randomly from the interval $[R_{min},R_{max}]$ and by choosing ϕ_i randomly from the interval $[0,\pi]$. In this study R_{min} and R_{max} were arbitrarily choosen to be one and eight, respectively.
2. Each x_i was strained to a known axial ratio (R_{act}) using the standard equations for passively straining an ellipse (Ramsay, 1967, Eqs. 5.22 and 5.27; Lisle, 1977b; Mulchrone and Meere, 2001).
3. For each known strain (R_{act}) 100 random initial datasets (x_i) were generated and strained, and the strain ellipse parameters (R_{calc},ϕ_{calc}) calculated for each x_i, using the method developed here.
4. The mean and standard deviation of the 100 (R_{calc},ϕ_{calc}) were used to estimate the sampling distribution for each R_{act}.

Simulations for $n = 100$ and $n = 400$ were carried out for a range of known strains (1–4.8 in 0.2 steps for $n = 100$ and 1–5 in unit steps for $n = 400$, see Figs. 4–6) similar to those routinely encountered in practice. For each known strain the true standard errors for the estimated average strain ellipse parameters (R_{calc},ϕ_{calc}) were obtained by analysis of 100 simulated data sets. The reliability of the Bootstrap was examined by comparing the bootstrap standard errors for one synthetic data set with the true standard errors calculated from 100 datasets.The meaning of the standard error or standard error of the mean depends on the context in which it is used. If there are multiple samples available from a single population then the standard error is the standard deviation of the sample means. However, if there is only one sample available from a population of size n, as is usually the case, the standard error for the mean of the population $(\bar{X})$ is estimated by:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

where σ is the standard deviation of the sample (Devore, 1995, p. 257). For sufficiently large n, this result is independent of the underlying probability distribution because of the central-limit theorem (Devore, 1995, p. 232), however, this only applies to simple statistics such as calculating the mean. Due to the complexity of R_s and ϕ_s, deriving an analytical expression for the standard error is intractable, hence the bootstrap approach is advocated.

The results for the set of configurations examined are shown in Figs. 4–6. The root mean squared percent deviations are between the true and bootstrap standard errors of 13.9 and 17.8% for the R_s and ϕ_s parameters. Surprisingly, the deviations are remarkably systematic—in the case of R_s, the bootstrap standard errors are 13.3% larger on average, whereas they are 17.5% lower for ϕ_s. In practical terms the discrepancies of the bootstrap standard errors from the true are not substantial and it is clear that the bootstrap standard errors can be used as a reasonable guide to the level of the uncertainty in the calculated strain estimates. For example, in Fig. 5d and f it is clear that the bootstrap confidence interval consistently brackets the true values of both R_s and ϕ_s.

4.2. Application to real data

The performance of the new method using the Bootstrap

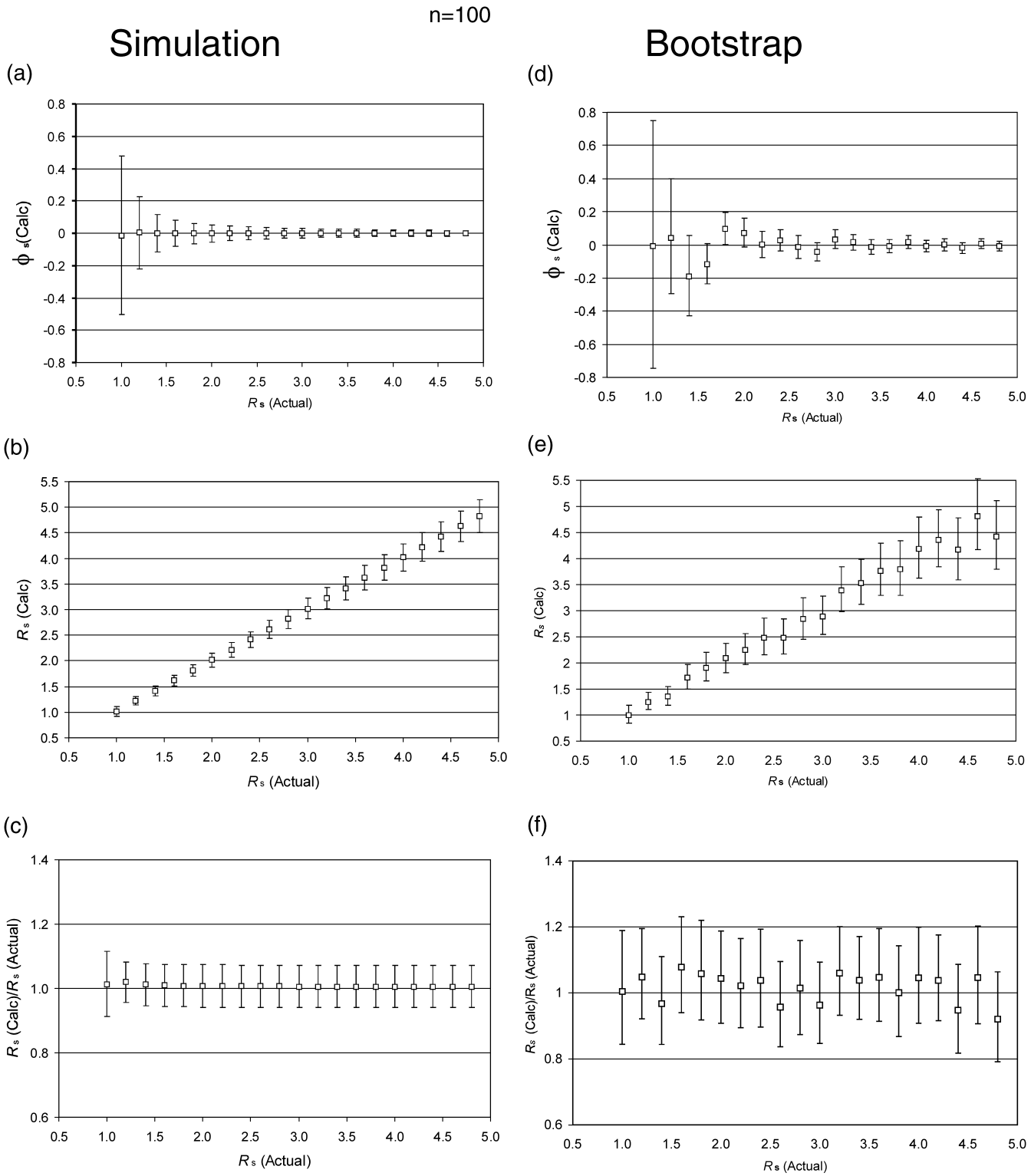

Fig. 5. Comparison of simulation (a)–(c) and bootstrap (d)–(f) for 100 ellipses per data set, axial ratio of the strain ellipse varies from 1 to 4.8 in 0.2 steps. Note that even though values calculated with the method presented here do not coincide with the true values, the confidence interval calculated with the bootstrap always includes the true value. Less variation is present in the simulation data because each single value is the average of 100 simulations. Notice also that the bootstrap confidence intervals are conservative in so far as they are always larger than the confidence intervals estimated from the simulation. Error bars are at the 95% confidence interval.

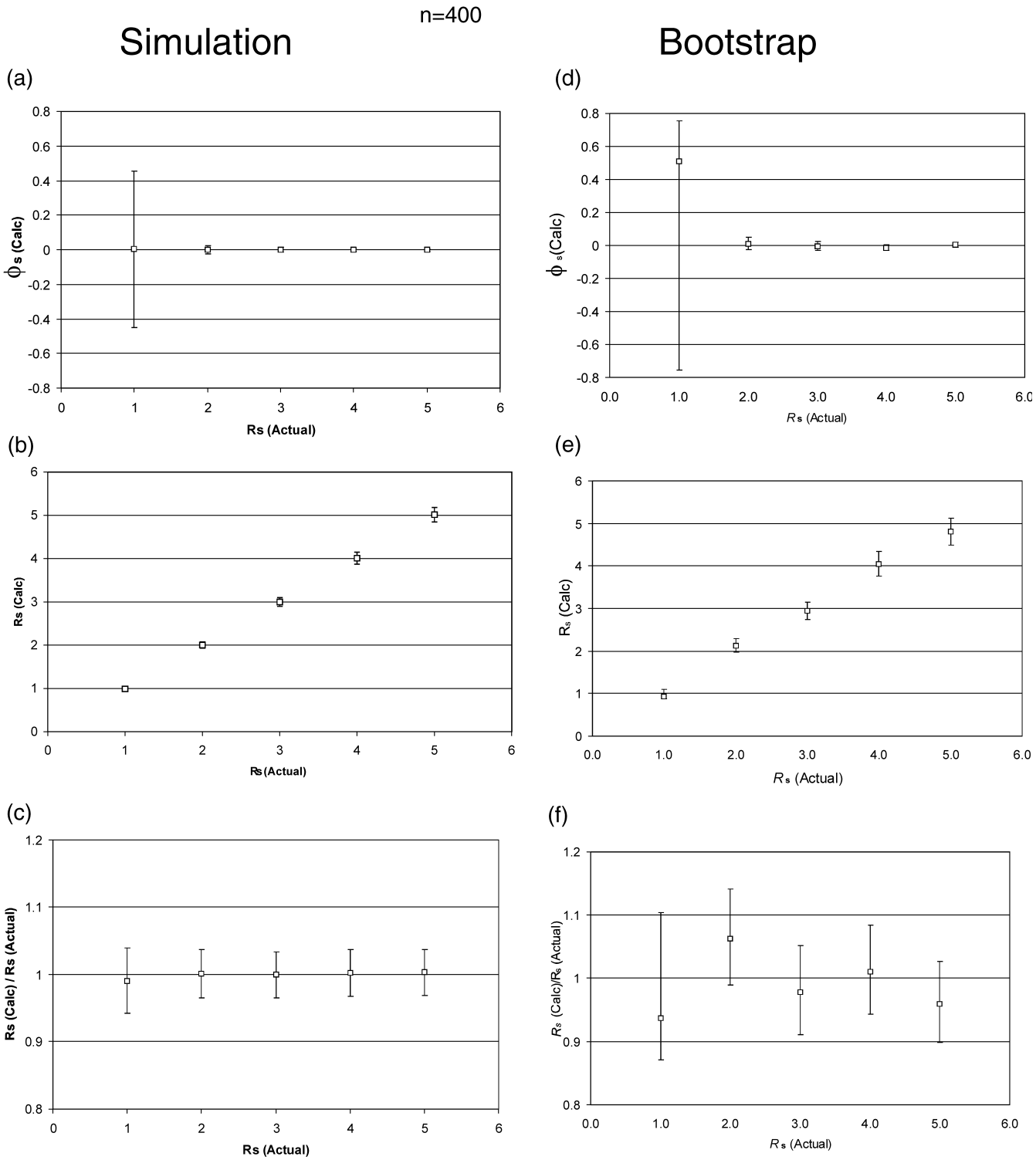

Fig. 6. Comparison of simulation (a)–(c) and bootstrap (d)–(f) for 400 ellipses per data set, axial ratio of the strain ellipse varies from 1 to 5 in unit steps.

approach was tested using real, previously published data. Twenty-three data sets were measured from thin sections of sandstones from the Variscides of southwest Ireland (Meere, 1995). The first seven samples are of sandstones that exhibit no observable tectonic fabric and were collected from north of the Irish Variscan cleavage front. The remaining 16 samples are of cleaved sandstones from south of the deformation front. Thin sections utilised in this study were cut orthogonal to the regional tectonic fabric and parallel to the fabric strike, i.e. the ab plane of Meere (1995). This ensured that the short axis of the calculated strain ellipse was a crude first approximation to the z-axis of the

Table 1
Results of analysis of the quartz clast data of Meere (1995) including skewness and calculated R_s values for Peach and Lisle (1979), Mulchrone and Meere (2001), Robin (1977) and present method. Lower and upper values for confidence intervals are included in brackets except for Peach and Lisle (1979) because confidence intervals were not recorded in the original analysis of Meere (1995)

Section	Skewness	Peach and Lisle (1979)	Mulchrone and Meere (2001)	Robin (1977)	This study
1ab	0.09	1.2	1.05 (1.00,1.15)	1.09 (1.01, 1.21)	1.10 (1.03, 1.22)
3ab	0.21	1.6	1.25 (1.10, 1.55)	1.33 (1.18, 1.53)	1.33 (1.16, 1.52)
7ab	0.50	1.3	1.25 (1.00, 1.45)	1.17 (1.05, 1.33)	1.17 (1.06, 1.33)
9ab	−0.03	1.8	1.85 (1.55, 2.05)	1.66 (1.52, 1.88)	1.67 (1.53, 1.89)
10ab	0.01	1.3	1.25 (1.10, 1.35)	1.35 (1.24, 1.51)	1.35 (1.23, 1.51)
11ab	0.21	1.7	1.40 (1.30, 1.55)	1.43 (1.28, 1.61)	1.44 (1.28, 1.63)
12ab	−0.08	1.1	1.10 (1.05, 1.15)	1.16 (1.06, 1.31)	1.17 (1.07, 1.31)
13ab	−0.79	1.4	1.30 (1.20, 1.35)	1.26 (1.13, 1.41)	1.28 (1.17, 1.41)
14ab	−0.50	1.7	1.30 (1.10, 1.45)	1.27 (1.13, 1.40)	1.27 (1.14, 1.40)
15ab	−0.20	1.5	1.60 (1.55, 1.70)	1.54 (1.38, 1.73)	1.53 (1.40, 1.72)
16ab	0.62	1.4	1.25 (1.20, 1.30)	1.20 (1.11, 1.33)	1.20 (1.12, 1.33)
17ab	−0.26	1.2	1.50 (1.35, 1.60)	1.53 (1.41, 1.68)	1.53 (1.41, 1.69)
18ab	−0.22	2	1.35 (1.10, 1.55)	1.26 (1.11, 1.42)	1.24 (1.10, 1.41)
19ab	−0.95	1.4	1.55 (1.30, 1.75)	1.58 (1.47, 1.75)	1.58 (1.47, 1.75)
21ab	0.64	1.1	1.35 (1.30, 1.45)	1.35 (1.23, 1.50)	1.35 (1.23, 1.51)
23ab	−0.54	1.5	1.40 (1.25, 1.50)	1.36 (1.24, 1.53)	1.39 (1.27, 1.55)

finite strain ellipsoid. Fifty R_f/ϕ readings were measured in each thin section using individual quartz clasts as strain markers. Clasts that best approximated an ellipse were used in preference to irregular, polycrystalline clasts.

Principal directions and finite strain ellipses were calculated using the THETA FORTRAN IV programme (Peach and Lisle, 1979) which utilises the 'Theta Curve' method of Lisle (1977b). The orientation of the principal stretching axis of the finite strain ellipse is estimated by the median orientation of the data. Furthermore, this method involves incrementally unstraining the R_f/ϕ data by a coaxial strain with stretching axis perpendicular to the preferred orientation of the data. After each increment of unstraining, a χ^2 test is applied to evaluate how well the orientations of the strain-modified data fit to a uniform circular distribution. The strain that produces the best fit to a uniform distribution (i.e. generates the lowest χ^2 value) is taken to be the best strain estimate. Recently Mulchrone and Meere (2001) presented an updated implementation of the minimum χ^2 method just described and the data have also been analysed using their program. Mulchrone and Meere (2001) estimate the orientation of the principal strain axes by the vector mean of the deformed marker orientations whilst Eqs. 5.22 and 5.27 of Ramsay (1967) are used for destraining purposes. In addition, the data were analysed using the Robin (1977) method, which involves calculating the logarithmic average of lines drawn on the strain markers parallel to the estimated principal finite strain directions of the sample in question.

In an attempt to further filter out populations whose orientations deviate from an initial uniform distribution, a fundamental prerequisite of the R_f/ϕ method, the skewness of the ϕ distribution for the original 23 samples was measured. ϕ distributions are not expected to exhibit significant skewness if all the prerequisites are satisfied (Borradaile, 1987). For the purposes of this study any distribution with a skewness value outside the range of $(-1 < x < 1)$ was rejected. High skewness values may indicate an initial preferred orientation of clasts, e.g. due to sedimentary imbrication, a complex deformation history or both.

Of the original 23 samples analysed in Meere (1995), seven are now deemed unsuitable for analysis because of their high skewness values. The results from the remaining 16 sections are presented in Table 1 and Figs. 7 and 8. When the 95% confidence interval error bars for the new method is included it is clear that only half the Peach and Lisle (1979) results fall within these error bars (Fig. 7a). The correlation between the two methods, with an 'r' value of 0.3, is consequently quite poor (Fig. 8a). There are several reasons for this lack of correlation. Firstly and most importantly, the implementation of this algorithm used by Meere (1995) was probably in error. Other less important factors may be that the Peach and Lisle (1979) method is only as accurate as the step size used in de-straining the sample and furthermore this method uses grouped, rather than the actual primary data to calculate χ^2. In general, any technique that uses all available data is preferable to a method using only a subset of the data and is almost always the most efficient statistic possible.

When the new method is compared with the Mulchrone and Meere (2001) method, which is also based on the 'Theta Curve' method of Lisle (1977b), there is a better correlation, an 'r' value of 0.7 (Figs. 7b and 8b), between results from the two methods. Consulting the confidence intervals given in Table 1, it is clear that the estimates from both Mulchrone and Meere (2001) and the present study overlap in almost all cases and thus they agree to within the precision of the methods. This also indicates that it is the implementation of Meere (1995) and not the 'Theta Curve' method, which

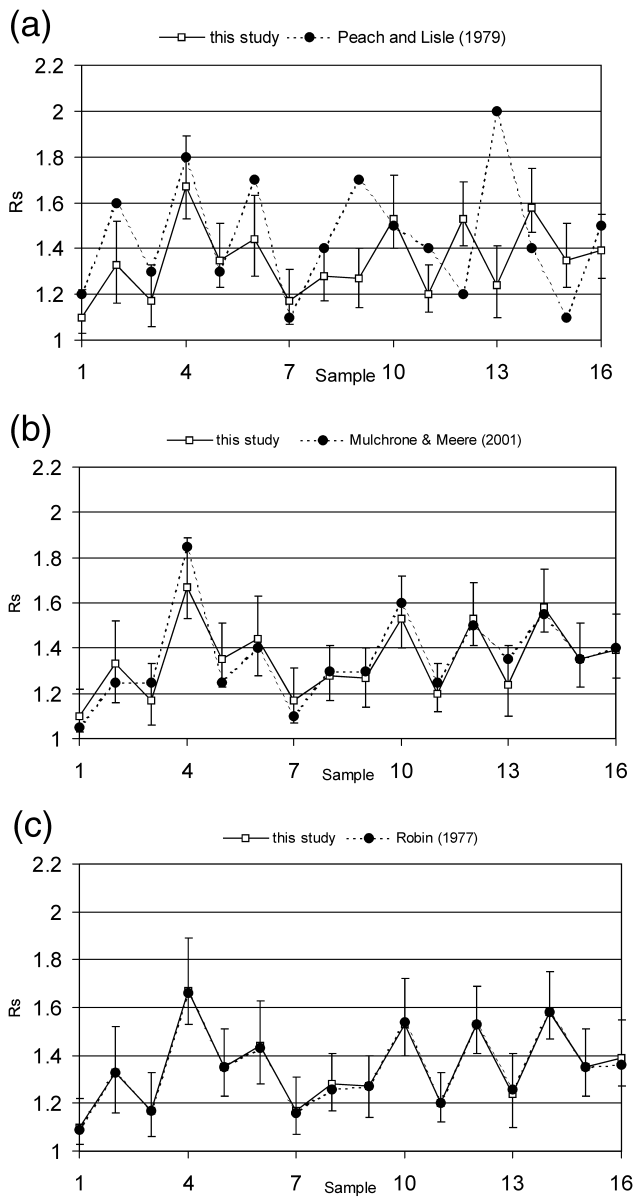

Fig. 7. A comparison of the new method with existing methods by the analysis of sandstone quartz clast data from the Irish Variscides. (a) A poor correlation with the method of Peach and Lisle (1979). (b) A good correlation with Mulchrone and Meere (2001) and an excellent correlation with the method of Robin (1977).

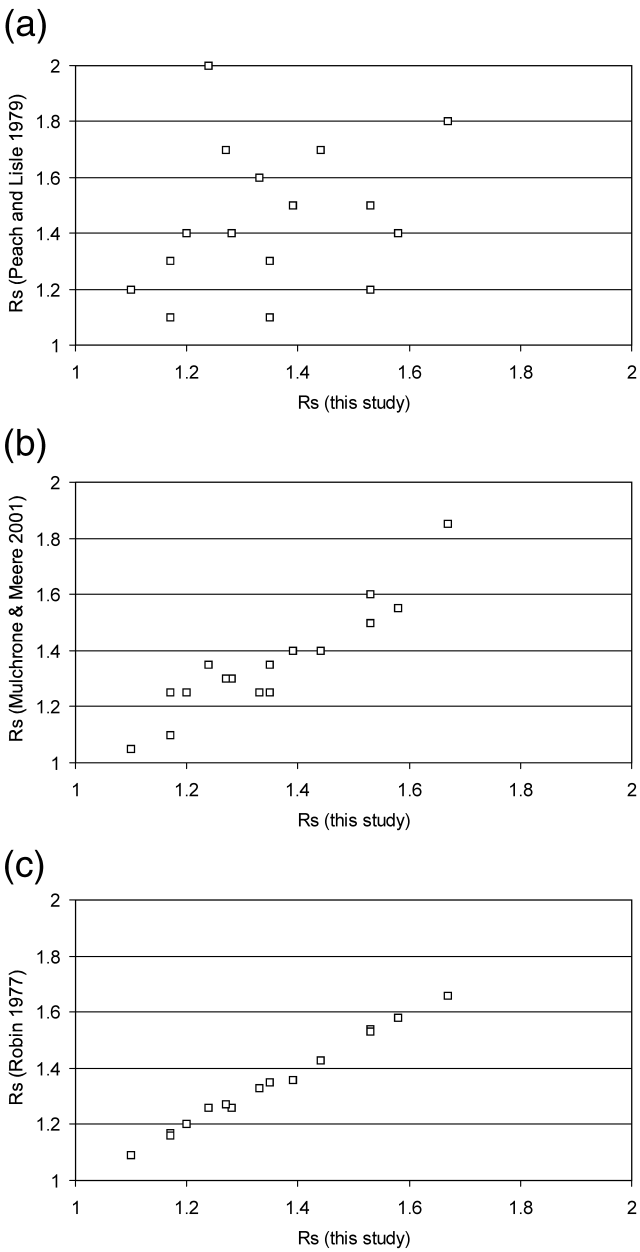

Fig. 8. Plots of R_s calculated using existing methods against R_s calculated using the method of this study. Ideally a straight line through the origin is optimal. Again Robin (1977) gives the best correlation.

resulted in the poor correlation between Peach and Lisle (1979) and the present method.

The Robin method exhibits an excellent correlation ('r' = 0.9) with the new method (Figs. 7c and 8c). It also has to be noted that the error bar range for the results of the new method are generally quite small allowing in this case a meaningful comparison of strain estimates and patterns in a relatively low strain environment. The overall pattern of low strain estimates noted by Meere (1995) north of the cleavage front is again evident and reinforced in the case of Robin (1977), Mulchrone and Meere (2001) and the new method.

5. Conclusions

A new method for calculating finite strain from distributions of elliptical objects is presented. Basic assumptions include that the initial distribution of objects is uniformly random with respect to orientation and that the distribution of axial ratios is independent of orientation. However, unlike many other approaches (Matthews et al., 1974; Borradaile, 1976; Robin, 1977) an estimate of the orientation of the long axis of the strain ellipse need not be independently determined prior to application of the method developed here. The method is based on the conceptually

simple fact that the mean radial length of a set of randomly oriented ellipses in the unstrained state evaluates to a circle, so that after strain the mean radial length equates to the strain ellipse. The mathematics associated with the method are relatively simple, so much so that the method could very easily be implemented in a spreadsheet application.

It is demonstrated that errors associated with the method are calculable using the bootstrap, and a comparison of the standard error of the bootstrap distribution with the standard error of simulated data indicates that the bootstrap errors are accurate. The method was applied to a set of real data from the Irish Variscides and results were cross-checked with those from the Robin (1977) method. There is a remarkable consistency between the results of both analyses.

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References

Arfken, G.B., Weber, H.J., 1995. Mathematical Methods for Physicists, Academic Press, San Diego.

Batschelet, E., 1981. Circular Statistics in Biology, Academic Press, London.

Borradaile, G.J., 1976. A study of a granite/granite–gneiss transition and accompanying schistosity formation in SE Spain. Journal of the Geological Society, London 132, 417–428.

Borradaile, G.J., 1987. Analysis of strained sedimentary fabrics: review and test. Canadian Journal of Earth Sciences 24, 442–455.

Borradaile, G.J., McArthur, J., 1991. Tests of strain analysis by experimental deformation. Tectonophysics 185, 325–333.

De Paor, D.G., 1980. Some limitations of the R_f/ϕ technique of strain analysis. Tectonophysics 64, T29–T31.

Devore, J.L., 1995. Probability and Statistics for Engineering and the Sciences, Duxbury Press, Pacific Grove.

Dunnet, D., 1969. A technique of finite strain analysis using elliptical particles. Tectonophysics 7, 117–136.

Dunnet, D., Siddans, A.W.B., 1971. Non-random sedimentary fabrics and their modification by strain. Tectonophysics 12, 307–325.

Efron, B., 1979. Bootstrap methods: another look at the jackknife. Annals of Statistics 7, 1–26.

Efron, B., Tibshirani, R.J., 1993. An Introduction to the Bootstrap, Chapman and Hall, London.

Elliott, D., 1970. Determination of finite strain and initial shape from deformed elliptical objects. Geological Society of America Bulletin 81, 2221–2236.

Fisher, N.I., 1993. Statistical Analysis of Circular Data, Cambridge University Press, Cambridge, UK.

Fraleigh, J.B., 1990. Calculus with Analytic Geometry, Addison-Wesley, Reading, MA.

Le Theoff, B., 1979. Non-coaxial deformation of elliptical markers. Tectonophysics 53, T7–T13.

Lisle, R.J., 1977a. Estimation of the tectonic strain ratio from the mean shape of deformed elliptical markers. Geologie en Mijnbouw 56, 140–144.

Lisle, R.J., 1977b. Clastic grain shape and orientation in relation to cleavage from the Aberystwyth Grits, Wales. Tectonophysics 39, 381–395.

Mardia, K.V., 1972. Statistics of Directional Data, Academic Press, London.

Matthews, P.E., Bond, R.A.B., Van Den Berg, J.J., 1974. An algebraic method of strain analysis using elliptical markers. Tectonophysics 24, 31–67.

Meere, P.A., 1995. The structural evolution of the western Irish Variscides: an example of obstacle tectonics? Tectonophysics 246, 97–112.

Mulchrone, K.F., Meere, P.A., 2001. A Windows program for the analysis of tectonic strain using deformed elliptical markers. Computers and Geosciences 27, 1253–1257.

Peach, C.J., Lisle, R.J., 1979. A Fortran IV program for the analysis of tectonic strain using deformed elliptical markers. Computers and Geosciences 5, 325–334.

Ramsay, J.G., 1967. Folding and Fracturing of Rocks, McGraw-Hill, New York.

Robin, P.F., 1977. Determination of geologic strain using randomly oriented strain markers of any shape. Tectonophysics 42, T7–T16.

Robin, P.F., Torrance, J.G., 1987. Statistical analysis of the effect of sample size on paleostrain calculation. I. Single face measurements. Tectonophysics 138, 311–317.

Shimamoto, T., Ikeda, Y., 1976. A simple algebraic method for strain estimation from deformed eillipsoidal objects. 1. Basic theory. Tectonophysics 36, 315–337.

Yu, H., Zheng, Y., 1984. A statistical analysis applied to the R_f/ϕ method. Tectonophysics 110, 151–155.